离散数学（011122）

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- **2.1 Basic Concepts of Propositional Logic**
- 2.2 Propositional logic equivalence transformations
- 2.3 Logical normal form

2.1.1 Propositions and Connectives

- Propositions and Truth Values (Simple Propositions, Compound Propositions)
- Connectives (¬, ∧, ∨, →, ↔)

2.1.2 Propositional Formulas and Their Classification

- Propositional Formulas and Their Assignments
- Truth Tables
- Classification of Propositional Formulas

- **Proposition**: A statement that can be judged as true or false.
- **Truth value of a proposition**: The result of the judgment, either true or false.
- **True proposition**: A proposition with a truth value of true.
- **False proposition**: A proposition with a truth value of false.

Note: Exclamatory sentences, imperative sentences, and interrogative sentences are not propositions. Also, paradoxes in declarative sentences or those with indeterminate judgment results are not propositions.

e.g. Example: Which of the following sentences are propositions?

(1) The capital of the People's Republic of China is Beijing. True proposition

(2) 2 + 3 ＝6. False proposition

(3) x + y ＞ 8. Undecided True Value

(4) Can you play tennis? Interrogative sentences

(5) There is life on planets other than Earth.. Indeterminate judgment results

(6) This ！ Exclamatory sentences

(7) Please close the door! Imperative sentences

(8) I am lying. Paradoxes

(1), (2), (5) are propositions. (3), (4), (6)~(8) are not propositions.

- **Simple Proposition (Atomic Proposition):** A proposition formed by simple statements.
  - **Symbolization of Simple Propositions:** Symbolization of simple propositions: Represented by p, q, r, …, p_i, q_i, r_i (where $i \geq 1$). '**1**' represents true, and '**0**' represents false.
- **Compound Proposition:**
  A statement formed by connecting simple propositions with logical connectives.
  Examples:

e.g.

(1) If the weather is good tomorrow, we will play ball.
Let p: The weather will be good tomorrow, and q: We will play ball. **If** p**, then** q.

(2) Luffy is drinking milk tea and scrolling through his phone.
Let p: Luffy is drinking milk tea, and q: Luffy is scrolling through his phone. p **and** q.

- **Definition 2.1:**"Non-*p*" (or "the negation of *p*") is called the ***negation*** of *p*, denoted as **¬*p***. The symbol ¬ is the negation connective, and it is defined such that **¬*p*** is true if and only if p is false.

  Example:
  - ***p***: 2 is a composite number. **¬*p***: 2 is not a composite number.
  - Since 2 is actually a prime number, ***p*** is false, and therefore **¬*p*** is true.

- **Definition 2.2:**

  "***p*** and ***q***" (or "***p*** with ***q***") is called the ***conjunction*** of p and q, denoted as ***p*∧*q***. The symbol ∧ is the conjunction connective, and it is defined such that ***p*∧*q*** is true if and only if both ***p*** and ***q*** are true simultaneously.

  Example:
  - ***p***: 2 is an even number. ***q***: 2 is a prime number.
  - ***p*∧*q***: 2 is an even prime number.
  - Since 2 is indeed both even and prime,

  both ***p*** and ***q*** are true, so ***p*∧*q*** is also true

Symbolize the following propositions.

(1) Wang Xiao is smart and hardworking.

(2) Wang Xiao is not only smart but also hardworking.

(3) Wang Xiao is smart, but not hardworking.

(4) Wang Xiao is not unintelligent, but rather not hardworking.

(5) Both Zhang Hui and Wang Li are outstanding students.

(6) Zhang Hui and Wang Li are classmates

(1) $p \wedge q$

(Let ***p***: Wang Xiao is smart, ***q***: Wang Xiao is hardworking.)

(2) $p \wedge q$

(3) $p \wedge \neg q$

(4) $\neg(\neg p) \wedge \neg q$

(5) $r \wedge s$

(Let *r*: Zhang Hui is an outstanding student, *s*: Wang Li is an outstanding student.)

(6) *t* (A simple proposition, "and" connects two nouns, the entire sentence is a simple proposition.)

- **Definition 2.3: "*p* or *q*"** is called the disjunctive form of ***p*** and ***q***, denoted as ***p***∨***q***. The symbol ∨ is called ***the Disjunction connective***, and ***p***∨***q*** is false if and only if both ***p*** and ***q*** are false.

e.g. Example: Wang Yan has studied English or French.

Let ***p***: Wang Yan has studied English,

q: Wang Yan has studied French.

Symbolized as ***p***∨***q***.

- The English term for "∨" is "inclusive" (Inclusive), corresponding to the "or" in everyday language. The English term for "∧" is "exclusive" (Exclusive), corresponding to the "and" in everyday language. "**Inclusive or**" and "**exclusive or**" are their combined forms.

e.g. Example: This task is to be done by either Zhang San or Li Si. Let $\boldsymbol{p}$: Zhang San does this task, $\boldsymbol{q}$: Li Si does this task. It should be symbolized as:

(1) $(p \wedge \neg q) \vee (\neg p \wedge q)$.

(2) $p \oplus q$

(3) P XOR q

e.g.»» Example: Symbolize the following propositions:

(1) 2 is a prime number or 4 is a prime number.

(2) 2 is a prime number or 3 is a prime number.

(3) 4 is a prime number or 6 is a prime number.

Solve: Let:

p: 2 is a prime number,
q: 3 is a prime number,
r: 4 is a prime number,
s: 6 is a prime number.

(1) $p \vee r$, $1 \vee 0 = 1$

(2) $p \vee q$, $1 \vee 1 = 1$

(3) $r \vee s$, $0 \vee 0 = 0$

e.g. Example: Symbolize the following propositions:

(4) Yuan Yuan can take an apple or a pear.

Solve: Let:

t: Yuan Yuan takes an apple,

u: Yuan Yuan takes a pear.

$(t \land \neg u) \lor (\neg t \land u)$

(5) Wang Xiaohong was born in 1975 or 1976

Solve: Let:

v: Wang Xiaohong was born in 1975,

w: Wang Xiaohong was born in 1976.

$(v \land \neg w) \lor (\neg v \land w)$

Can logical expression "$(v \land \neg w) \lor (\neg v \land w)$" be converted to "$v \lor w$"?

■ **Definition 2.4**

"If ***p***, then ***q***" is called the implication of ***p*** and ***q***, denoted as ***p***→***q***. ***p*** is called the antecedent (or hypothesis) of the implication, and ***q*** is called the consequent (or conclusion). The symbol → is called the ***implication connective***.

- It is defined that ***p***→***q*** is false if and only if p is true and q is false.

e.g. **Example:**

"If the weather is good tomorrow, we will go on an outing."

Let ***p***: The weather is good tomorrow,

q: We will go on an outing.

This can be formalized as ***p***→***q***.

- **Logical relationship of *p*→*q*:**
  q is a necessary condition for ***p***, and ***p*** is a sufficient condition for ***q***.
- **Various ways to express "If *p*, then *q*" (all having the same truth value as *p*→*q*):**
  (1) If ***p***, then ***q***.
  (2) If ***p***, just ***q***. (Emphasizes that the existence of ***p*** is a necessary condition for the existence of ***q***.)
  (3) ***p*** only if ***q***. (Means that for ***p*** to be true, ***q*** must also be true.)
  (4) Only if ***q***, then ***p***. (Indicates that the truth of ***q*** is a necessary condition for the truth of ***p***.)
  (5) Unless ***q***, not ***p***. (If ***q*** is not the case (***q*** is false), then ***p*** does not hold (***p*** is false).

Truth and Falsity of the Implication $p \to q$:

- When p is true, q must also be true for $p \to q$ to be "**true**".
- When p is true and q is false, $p \to q$ is "**false**".
- When p is false, the implication does not impose any restrictions on the truth value of q, and $p \to q$ is "**true**".

e.g. **Example:**

Let p be defined as "It is raining today," and q be defined as "The ground is wet." The implication $p \to q$ can be described as "If it rains today, then the ground will be wet."

- In this case, the implication is **only false** if it rains (p is true) and the ground is not wet (q is false). In all other scenarios (such as it not raining, or it raining and the ground being wet), the implication is true.

e.g. Example:

Let *p*: It is cold, Let *q*: Wang wears a down jacket.,

Here's the translation of the propositions with logical symbolism:

(1) As long as it is cold, Xiao Wang will wear a down jacket. $p \to q$

(2) Because it is cold, Xiao Wang wears a down jacket. $p \to q$

(3) If Xiao Wang does not wear a down jacket, then it is not cold. $\neg q \to \neg p$ or $p \to q$

(4) Only if it is cold, Xiao Wang will wear a down jacket. $q \to p$

otherwise

e.g.»» Example:

Let ***p***: It is cold, Let ***q***: Wang wears a down jacket.

Here's the translation of the propositions with logical symbolism:

(5) Unless it is cold, Xiao Wang will wear a down jacket. $q \to p$

(6) Unless Xiao Wang wears a down jacket, otherwise it will not be cold. $p \to q$

(7) If it is not cold, then Xiao Wang will not wear a down jacket. $\neg p \to \neg q$ or $q \to p$

(8) Xiao Wang wears a down jacket only if it is cold. $q \to p$

■ **Definition 2.5:**

The statement "***p*** if and only if ***q***" is called the equivalence of ***p*** and ***q***, denoted by ***p*↔*q***, where ↔ is called the ***biconditional operator***. It is defined that ***p*↔*q*** is true if and only if both ***p*** and ***q*** are true or both are false.

■ **Logical Relationship of p↔q:**

p and ***q*** are mutually sufficient and necessary conditions for each other.

e.g.>>> **Example:**

Zhang San can do this task well, and only Zhang San can do it well.
Let ***p***: Zhang San does the task, ***q***: The task is done well.
This can be formalized as: ***p*↔*q***.

e.g. »» **Example:** Determine the truth value of the following compound propositions:

(1) 2 + 2 = 4 if and only if 3 + 3 = 6. 1

(2) 2 + 2 = 4 if and only if 3 is even. 0

(3) 2 + 2 = 4 if and only if the sun rises in the east. 1

(4) 2 + 2 = 5 if and only if the sun rises in the west. 0

(5) A necessary and sufficient condition for $f(x)$ to be differentiable at x_0 is that it is continuous at x_0. 0

Truth Values of Basic Compound Propositions

p	q	$p\wedge q$	$p\vee q$	$p\to q$	$p\leftrightarrow q$
0	0	0	0	1	1
0	1	0	1	1	0
1	0	0	1	0	0
1	1	1	1	1	1

Precedence of Logical Connectives:

Parentheses (), Negation (¬), Conjunction (∧), Disjunction (∨), Implication (→), Biconditional (↔).

- **Same Level:** Evaluated from left to right.

- **Propositional Constants:** Simple propositions
- **Propositional Variables:** Variables that can take the value 0 (true) or 1 (false).
- **Definition 2.6 Well-Formed Formula (Propositional Formula, Formula):** A well-formed formula is **recursively defined** as follows:

(1) A single propositional constant or variable is a well-formed formula, also called an atomic formula.

(2) If A is a well-formed formula, then (¬A) is also a well-formed formula.

(3) If A and B are well-formed formulas, then (A ∧ B), (A ∨ B), (A → B), and (A ↔ B) are also well-formed formulas.

(4) Only those expressions formed by a finite number of applications of (1) through (3) are considered well-formed formulas.

e.g. **Example: 0, p, $\neg p \vee q$, $(p \vee q) \wedge (\neg p \vee r)$, $p \vee q \to r$, $(p \to q) \to r$**

Definition 2.7

(1) A single propositional variable or propositional constant is a 0-layer formula.

(2) A formula A is an (n+1)-layer formula (where $n \geq 0$) if one of the following conditions is met:

① $A = \neg B$, where B is an n-layer formula.

② $A = B \wedge C$, where B and C are i-layer and j-layer formulas, respectively, and $n = \max(i, j)$.

③ $A = B \vee C$, where B and C are i-layer and j-layer formulas, respectively, and $n = \max(i, j)$.

④ $A = B \rightarrow C$, where B and C are i-layer and j-layer formulas, respectively, and $n = \max(i, j)$.

⑤ $A = B \leftrightarrow C$, where B and C are i-layer and j-layer formulas, respectively, and $n = \max(i, j)$.

e.g. **Example:** The propositional formulas

P	(0-layer)
$\neg p$	(1-layer)
$\neg p \rightarrow q$	(2-layer)
$(\neg(p \rightarrow q)) \leftrightarrow r$	(3-layer)
$((\neg p \wedge q) \rightarrow r)) \leftrightarrow (\neg r \vee s)$	(4-layer)

Definition 2.8

- Let $p_1,p_2,\ldots,p_n$ be all the propositional variables appearing in the formula *A*.
- Assigning a set of truth values to $p_1,p_2,\ldots,p_n$ is called an **assignment** or **interpretation** for *A*.
- An assignment that makes the formula true is called a ***satisfying assignment***, and an assignment that makes the formula false is called a ***falsifying assignment***.

Definition 2.8 Explanation:

(1) An assignment is denoted as $a=a_1a_2...a_n$, where each a_i is either 0 or 1, and the a_i are written without any punctuation marks between them.

(2) Generally, the assignment corresponds to the propositional variables in the order of their subscripts or alphabetical order. That is:

- When all propositional variables in *A* are $p_1,p_2,...,p_n$, assigning $a_1a_2...a_n$ to A means $p_1=a_1$, $p_2=a_2$,..., $p_n=a_n$.
- When all propositional variables in *A* are $p,q,r,...$, assigning $a_1a_2 a_3$ ... to A means $p_1=a_1$, $p_2=a_2$, $p_3=a_3$,... .

e.g. »» **Example:**

- Formula $A=(\neg p1\wedge\neg p2\wedge\neg p3)\vee(p1\wedge p2)$
  - The assignment 000 is a **satisfying assignment** (makes the formula **true**).
  - The assignment 001 is a **falsifying assignment** (makes the formula **false**).
- Formula $B=(p\rightarrow q)\rightarrow r$
  - The assignment 000 is a **falsifying assignment** (makes the formula **false**).
  - The assignment 001 is a **satisfying assignment** (makes the formula **true**).

- **Truth Table:** A list of the values taken by a propositional formula under all possible assignments.
  - A formula with n variables has 2^n assignments.

e.g. **Example:** Provide the truth table for the following propositional formula.

(1) $(q \to p) \wedge q \to p$

p q	$q \to p$	$(q \to p) \wedge q$	$(q \to p) \wedge q \to p$
0 0	1	0	1
0 1	0	0	1
1 0	1	0	1
1 1	1	1	1

e.g. **Example:** Provide the truth table for the following propositional formula.

(2) $\neg(\neg p \vee q) \wedge q$

p q	$\neg p$	$\neg p \vee q$	$\neg(\neg p \vee q)$	$\neg(\neg p \vee q) \wedge q$
0 0	1	1	0	0
0 1	1	1	0	0
1 0	0	0	1	0
1 1	0	1	0	0

e.g. **Example:** Provide the truth table for the following propositional formula.

(3) $(p \vee q) \to \neg r$

p q r	$p \vee q$	$\neg r$	$(p \vee q) \to \neg r$
0 0 0	0	1	1
0 0 1	0	0	1
0 1 0	1	1	1
0 1 1	1	0	0
1 0 0	1	1	1
1 0 1	1	0	0
1 1 0	1	1	1
1 1 1	1	0	0

- **Tautology (Always True):** A propositional formula that is never false under any assignment.
- **Contradiction (Always False):** A propositional formula that is never true under any assignment.
- **Satisfiable Formula:** A propositional formula that is not a contradiction.

  Note: A tautology is satisfiable, but the converse is not true.

Examples:

e.g. (1) $(q \to p) \wedge q \to p$ is a tautology.

(2) $\neg(\neg p \vee q) \wedge q$ is a contradiction.

(3) $(p \vee q) \to \neg r$ is a satisfiable formula that is not a tautology.